

Title	A KINETIC APPROACH TO A COMPARISON THEOREM FOR DEGENERATE PARABOLIC EQUATIONS(Dynamics of functional equations and numerical simulation)
Author(s)	Kobayashi, Kazuo
Citation	数理解析研究所講究録 (2006), 1475: 155-166
Issue Date	2006-02
URL	http://hdl.handle.net/2433/48194
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

A KINETIC APPROACH TO A COMPARISON THEOREM FOR DEGENERATE PARABOLIC EQUATIONS

早稲田大学・教育学部 小林和夫 (Kazuo Kobayasi)

Department of Mathematics, School of Education, Waseda University
e-mail: kzokoba@waseda.jp

1. STATEMENT OF THE RESULT.

Let Ω be an open bounded subset of \mathbb{R}^d and $T \in (0, +\infty]$. Let Q denote the set $(0, T) \times \Omega$, $\partial\Omega$ the boundary of Ω , $\mathbf{n}(\bar{x})$ the outward unit normal to Ω at a point $\bar{x} \in \partial\Omega$ and Σ the set $(0, T) \times \partial\Omega$. We consider the following parabolic-hyperbolic problem:

$$\partial_t u + \operatorname{div} A(u) - \Delta \beta(u) = 0 \quad \text{in } Q \quad (1.1)$$

with the initial condition:

$$u(0, x) = u_0(x), \quad x \in \Omega, \quad (1.2)$$

and the boundary condition:

$$u(t, x) = u_b(t, x), \quad (t, x) \in \Sigma, \quad (1.3)$$

where the flux function A belongs to $C^1(\mathbb{R})$ and the function β is non-decreasing and Lipschitz continuous. This monotonicity assumption of β allows us some degenerate diffusion cases which appear in many interesting models, for example, filtration problems in porous media [2,5,8].

In the nondegenerate case (in which the function β is strictly increasing), the problem (1.1) is of parabolic type and hence the existence and uniqueness of solutions are well known. In the case where $\beta' \equiv 0$, the problem (1.1) being a nonlinear hyperbolic problem, the uniqueness

of weak solutions is not ensured, and one must consider a notion of entropy solution, relying on the notion of boundary entropy-flux pairs to recover uniqueness (see [11,16]). When β is merely a nondecreasing function, in the case of homogeneous boundary data, i.e., $u_b \equiv 0$, Carrillo [3] succeeded in proving the uniqueness of entropy solutions by mainly using the dedoubling variable technique developed by Kružkov [11]. The equivalence of entropy solutions and weak solutions is also considered in [10]. In the case of nonhomogeneous boundary data existence and uniqueness of entropy solutions to (1.1)-(1.3) have been proved in [1,14,15]. The method used there is also the dedoubling variable technique.

On the other hand Perthame [12,17] proved the uniqueness of entropy solutions to the Cauchy problem of the conservation law (in which $\beta' \equiv 0$ and $\Omega = \mathbb{R}^d$) by using the kinetic formulation which is introduced by Lions, Perthame and Tadmor [12], without relying on the dedoubling variable technique. Imbert and Vovelle [9] developed analogous techniques for conservation laws with boundary conditions, proved the Comparison Theorem for entropy sub- and supersolutions, and applied their results to the BGK-like model. This technique was also applied in [6] to study the parabolic approximation of a multidimensional conservation law with initial and boundary conditions.

The purpose of this note is to give a comparison result for their sub- and supersolutions by using kinetic techniques. Although the L^1 contractivity and, therefore, uniqueness of entropy weak solutions has been obtained, it would seem that any comparison theorem for those solutions is not proven.

According to [14] we introduce the definition of entropy sub- and supersolution.

Define

$$\operatorname{sgn}^+(r) = \begin{cases} 1 & \text{if } r > 0, \\ 0 & \text{if } r \leq 0, \end{cases} \quad \text{and} \quad \operatorname{sgn}^-(r) = \begin{cases} -1 & \text{if } r < 0, \\ 0 & \text{if } r \geq 0, \end{cases}$$

and $r^\pm = \operatorname{sgn}^\pm(r)r$.

Definition 1.1. A function u of $L^1(Q)$ is said to be a weak solution of the problem (1.1) - (1.3) if it satisfies: $\beta(u) - \beta(u_b) \in L^2(0, T; H_0^1(\Omega))$, $A(u) \in L^1(Q)^d$ and

$$\int_Q u \varphi_t + (A(u) - \nabla \beta(u)) \cdot \nabla \varphi dx dt + \int_\Omega u_0 \varphi(0, x) dx = 0 \quad (1.4)$$

for any $\varphi \in C_c^\infty([0, T] \times \Omega)$.

Definition 1.2. Let $u \in L^\infty(Q)$. u is said to be an entropy subsolution of (1.1) - (1.3) if it is a weak solution and satisfies:

$$\begin{aligned} & \int_Q (u - \kappa)^+ \partial_t \varphi + (\mathcal{F}^+(u, \kappa) - \nabla(\beta(u) - \beta(\kappa))^+) \cdot \nabla \varphi dx dt \\ & + \int_\Omega (u_0 - \kappa)^+ \varphi(0, x) dx + M \int_\Sigma (u_b - \kappa)^+ \varphi d\sigma dt \geq 0 \end{aligned} \quad (1.5)$$

for any $\kappa \in \mathbb{R}$ and any $\varphi \in C_c^\infty([0, T) \times \mathbb{R}^d)^+$ such that $\text{sgn}^+(\beta(u_b) - \beta(\kappa))\varphi = 0$ a.e. on Σ .

u is said to be an entropy supersolution if (1.7) is replaced by

$$\begin{aligned} & \int_Q (u - \kappa)^- \partial_t \varphi + (\mathcal{F}^-(u, \kappa) - \nabla(\beta(u) - \beta(\kappa))^-) \cdot \nabla \varphi dx dt \\ & + \int_\Omega (u_0 - \kappa)^- \varphi(0, x) dx + M \int_\Sigma (u_b - \kappa)^- \varphi d\sigma dt \geq 0 \end{aligned} \quad (1.6)$$

for any $\kappa \in \mathbb{R}$ and any $\varphi \in C_c^\infty([0, T) \times \mathbb{R}^d)^+$ such that $\text{sgn}^-(\beta(u_b) - \beta(\kappa))\varphi = 0$ a.e. on Σ . Here $C_c^\infty([0, T) \times \mathbb{R}^d)^+$ is the set of nonnegative functions in $C_c^\infty([0, T) \times \mathbb{R}^d)$.

We also set

$$M = \sup\{|A'(r)|; |r| \leq \max\{\|u_0\|_{L^\infty(\Omega)}, \|u_b\|_{L^\infty(\Sigma)}\}\} \quad (1.7)$$

and

$$L = \max_{1 \leq i \leq N} \|\Delta_{\bar{x}} h_i(\overline{T_i x})\|_{L^\infty(\Sigma_{\lambda_i})}. \quad (1.8)$$

We are now in a position to state the main theorem which obviously extends the L^1 contractive property for entropy solutions

Theorem Assume that the following conditions hold:

(A1) Ω is a bounded open subset of \mathbb{R}^d whose boundary $\partial\Omega$ is C^2 , $A \in C^1(\mathbb{R}, \mathbb{R})$ and $\beta : \mathbb{R} \rightarrow \mathbb{R}$ is a nondecreasing Lipschitz continuous function.

(A2) $u_0 \in L^\infty(\Omega)$ and $u_b \in L^\infty(\Sigma)$.

Let $u \in L^\infty(Q)$ be an entropy subsolution of (1.1) - (1.3) with data (u_0, u_b) and let \tilde{u} be an entropy supersolution of (1.1) - (1.3) with data $(\tilde{u}_0, \tilde{u}_b)$. Then we have

$$\begin{aligned} & \frac{1}{T} \int_0^T \int_\Omega (u(t, x) - \tilde{u}(t, x))^+ dx dt \\ & \leq \int_\Omega (u_0(x) - \tilde{u}_0(x))^+ dx + M \int_0^T \int_{\partial\Omega} (u_b(t, x) - \tilde{u}_b(t, x))^+ d\sigma dt \\ & + \frac{L}{2} \int_0^T \int_{\partial\Omega} (\beta(u_b(t, x)) - \beta(\tilde{u}_b(t, x)))^+ d\sigma dt. \end{aligned} \quad (1.9)$$

2. SKETCH OF PROOF.

The semi-Kruřkov entropies are the convex functions defined by

$$\eta_k^\pm(r) = (r - k)^\pm, \quad k \in \mathbb{R},$$

while the corresponding entropy flux are the function defined by

$$\mathcal{F}^\pm(r, k) = \operatorname{sgn}^\pm(r - k)(A(r) - A(k)).$$

For a function $u \in L^\infty(Q)$ and $\xi \in \mathbb{R}$ we set

$$f_\pm(t, x, \xi) = \operatorname{sgn}^\pm(u(t, x) - \xi).$$

We assume that Ω is a C^2 bounded open subset in \mathbb{R}^d . Thus, we can find a finite open cover $\{B_i\}_{i=0}^N$ of $\overline{\Omega}$ and a partition of unity $\{\lambda_i\}_{i=0}^N$ on $\overline{\Omega}$ subordinate to $\{B_i\}_{i=0}^N$ such that, for $i \geq 1$, up to a change of coordinates represented by an orthogonal matrix T_i , the set $\Omega \cap B_i$ is the epigraph of a C^2 function $h_i : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$, that is to say:

$$\Omega_{\lambda_i} \cap B_i = \{x \in B_i; (T_i x)_d > h_i(\overline{T_i x})\}$$

and

$$\partial\Omega_{\lambda_i} = \partial\Omega \cap B_i = \{x \in B_i; (T_i x)_d = h_i(\overline{T_i x}),$$

where $x = (\bar{x}, x_d) \in \mathbb{R}^d$ and $\bar{x} = (x_1, \dots, x_{d-1})$. For simplicity we will drop the index i and we suppose that the change of coordinates is trivial: $Y_i = Id$. We also write $Q_\lambda = (0, T) \times \Omega_\lambda$, $\Sigma_\lambda = (0, T) \times \partial\Omega_\lambda$, $\Pi_\lambda = \{\bar{x}; x \in \operatorname{supp}(\lambda) \cap \Omega\}$ and $\Theta_\lambda = (0, T) \times \Pi_\lambda$. We denote by $\mathbf{n}(\bar{x})$ the outward unit normal to Ω_λ at a point $(\bar{x}, h(\bar{x}))$ of $\partial\Omega_\lambda$ and by $d\sigma(\bar{x})$ the $(d-1)$ -dimensional area element in $\partial\Omega_\lambda$:

$$\mathbf{n}(\bar{x}) = (1 + |\nabla_{\bar{x}} h(\bar{x})|^2)^{-1/2} (\nabla_{\bar{x}} h(\bar{x}), -1),$$

$$d\sigma(\bar{x}) = (1 + |\nabla_{\bar{x}} h(\bar{x})|^2)^{1/2} d\bar{x}.$$

To regularize the functions, for small $\rho, s > 0$ let us consider a smooth function $\theta_{\rho,s} : \mathbb{R} \rightarrow \mathbb{R}^+$ such that $\operatorname{supp} \theta_{\rho,s} \subset [\rho s/2, (1+\rho)s]$, $\theta_{\rho,s}(r) = s^{-1}$ for $\rho s \leq r \leq s$ and $\int_{\mathbb{R}} \theta_{\rho,s}(r) dr = 1$. Then, for $\nu > 0$ and $\epsilon = (\epsilon_1, \dots, \epsilon_d) \in (\mathbb{R}^+)^d$, we set $\gamma_{\rho,\epsilon}(x) = \prod_{i=1}^d \theta_{\rho,\epsilon_i}(x_i)$ and $\gamma_{\rho,\nu,\epsilon}(t, x) = \theta_{\rho,\nu}(t) \gamma_{\rho,\epsilon}(x)$.

For simplicity, we will also use the following notations:

$$\mathbf{n}_1 = \sqrt{1 + |\nabla_{\bar{x}} h(\bar{x})|^2} \quad \mathbf{n},$$

$$\bar{x}_r = (\bar{x}, h(\bar{x}) + r) \quad \text{for } \bar{x} = (x_1, \dots, x_{d-1}),$$

K. Kobayasi

ψ^λ stands for $\psi\lambda$ and $\bar{\psi}$ denotes the restriction of ψ to $\Sigma \times \mathbb{R}_\xi$, i.e., $\bar{\psi}(t, \bar{x}, \xi) = \psi(t, \bar{x}, h(\bar{x}, \xi))$, where ψ is a function on $[0, T) \times \mathbb{R}^{d+1}$ and λ is an element of the partition of unity $\{\lambda_i\}_{i=0}^N$. Moreover we set $s \vee t = \max\{s, t\}$ and $s \wedge t = \min\{s, t\}$.

The proof of the theorem will follow from the following three lemmas whose proofs will be given in the forthcoming paper.

Lemma 2.1. *Let u be an entropy subsolution with data (u_0, u_b) and let λ be an element of the partition of unity $\{\lambda_i\}_{i=0}^N$. Then we have:*

(a) *There exists $f_+^{\tau_0} \in L^\infty(\Omega \times \mathbb{R})$ such that*

$$\lim_{s \rightarrow +0} \int_{\Omega \times \mathbb{R}} \left[\frac{1}{s} \int_0^s f_+(t, x, \xi) dt \right] \phi \, dx d\xi = \int_{\Omega \times \mathbb{R}} f_+^{\tau_0} \phi \, dx d\xi \quad (2.1)$$

for any $\phi \in C_c^\infty(\Omega \times \mathbb{R})$.

(b) *For any $\psi \in C_c^\infty([0, T) \times \mathbb{R}^{d+1})^+$ and any weak* cluster point f_+^τ of $\frac{1}{s} \int_0^s f_+(t, \bar{x}_\tau, \xi) d\tau$ as $s \rightarrow +0$ in $L^\infty(\Theta_\lambda \times \mathbb{R})$, we have*

$$\begin{aligned} & \int_{Q_\lambda \times \mathbb{R}} (f_+(\partial_t + a \cdot \nabla) \psi^\lambda - \beta' \nabla f_+ \cdot \nabla \psi^\lambda) dt dx d\xi \\ & + \int_{\Omega \times \mathbb{R}} f_+^{\tau_0} \psi^\lambda(0, x) dx d\xi + \int_{\Theta_\lambda \times \mathbb{R}} \beta' (\nabla_{\bar{x}} h(\bar{x}) \cdot \nabla_{\bar{x}} f_+^b) \bar{\psi}^\lambda \, dt d\bar{x} d\xi \\ & + \int_{\Theta_\lambda \times \mathbb{R}} (-\mathbf{n}_1 \cdot a) f_+^\tau \bar{\psi}^\lambda \, dt d\bar{x} d\xi \\ & \geq \int_{Q_\lambda \times \mathbb{R}} \partial_\xi \psi^\lambda d(m_+ + n_+). \end{aligned} \quad (2.2)$$

Lemma 2.2. *There exist families of probability measures $\{\nu_x^{\tau_0}\}_{x \in \Omega}$ and $\{\tilde{\nu}_x^{\tau_0}\}_{x \in \Omega}$ on \mathbb{R}_ξ , called Young measures, supported in $(-\infty, \|u\|_{L^\infty}]$ and $[-\|\tilde{u}\|_{L^\infty}, \infty)$, respectively, and nonnegative functions $m_+^0(x, \xi)$ and $\tilde{m}_-^0(x, \xi)$ defined on $\Omega \times \mathbb{R}_\xi$ such that*

$$\begin{aligned} & m_+^0, \tilde{m}_-^0 \in C(\mathbb{R}_\xi; w\text{-}\mathcal{M}^+(\Omega)), \\ & \lim_{\xi \rightarrow \infty} m_+^0(x, \xi) = \lim_{\xi \rightarrow -\infty} \tilde{m}_-^0(x, \xi) = 0 \quad \text{for a.e. } x \in \Omega, \\ & f_+^{\tau_0}(x, \xi) = \nu_x^{\tau_0}([\xi, \infty)) = \partial_\xi m_+^0(x, \xi) + \text{sgn}^+(u_0(x) - \xi) \end{aligned} \quad (2.3)$$

and

$$\tilde{f}_-^{\tau_0}(x, \xi) = -\tilde{\nu}_x^{\tau_0}((-\infty, \xi]) = \partial_\xi \tilde{m}_-^0(x, \xi) + \text{sgn}^-(\tilde{u}_0(x) - \xi).$$

Lemma 2.3. *Let λ be an element of the partition of unity $\{\lambda_i\}_{i=0}^N$ and let f_+^τ and \tilde{f}_-^τ be weak* cluster point of $\frac{1}{s} \int_0^s f_+(t, \bar{x}_r, \xi) dr$ and $\frac{1}{s} \int_0^s \tilde{f}_-(t, \bar{x}_r, \xi) dr$, respectively, as $s \rightarrow +0$, in $L^\infty(\Theta_\lambda \times \mathbb{R})$. There exist Young measures $\{\nu_{t,y}^\tau\}_{(t,y) \in \Sigma}$ and $\{\tilde{\nu}_{t,y}^\tau\}_{(t,y) \in \Sigma}$ on \mathbb{R}_ξ , supported in $(-\infty, \|u\|_{L^\infty}]$ and $[-\|\tilde{u}\|_{L^\infty}, \infty)$, respectively, and nonnegative functions $m_+^b(t, y, \xi)$ and $\tilde{m}_-^b(t, y, \xi)$ defined on $\Sigma \times \mathbb{R}_\xi$ such that*

$$\lim_{\xi \rightarrow \infty} m_+^b(t, y, \xi) = \lim_{\xi \rightarrow -\infty} \tilde{m}_-^b(t, y, \xi) = 0 \quad \text{for a.e. } (t, y) \in \Sigma.,$$

$$f_+^\tau(t, y, \xi) = \nu_{t,y}^\tau([\xi, \infty)), \quad \tilde{f}_-^\tau = -\tilde{\nu}_{t,y}^\tau((-\infty, \xi]),$$

$$(-a \cdot \mathbf{n}_1) f_+^\tau = \partial_\xi m_+^b + M \operatorname{sgn}^+(u_b - \xi) \quad (2.4)$$

$$(-a \cdot \mathbf{n}_1) \tilde{f}_-^\tau = \partial_\xi \tilde{m}_-^b + M \operatorname{sgn}^-(\tilde{u}_b - \xi),$$

$$\int_{\Theta_\lambda} m_+^b(t, \bar{x}_0, \xi) \bar{\varphi}^\lambda(t, \bar{x}_0) dt d\bar{x} \geq 0 \quad (2.5)$$

for any $\bar{\varphi} \in C(\Sigma)^+$ satisfying $\operatorname{sgn}^+(\beta(u_b) - \beta(\xi)) \bar{\varphi} = 0$ a.e. on Σ and

$$\int_{\Theta_\lambda} \tilde{m}_-^b(t, \bar{x}_0, \xi) \bar{\varphi}^\lambda(t, \bar{x}_0) dt d\bar{x} \geq 0$$

for any $\bar{\varphi} \in C(\Sigma)^+$ satisfying $\operatorname{sgn}^-(\beta(\tilde{u}_b) - \beta(\xi)) \bar{\varphi} = 0$ a.e. on Σ .

We continue the proof of Theorem. Let f_+, n_+ and m_+ be the functions defined for u as above. $f_+^{\tau_0}$ denotes the time kinetic traces and f_+^τ a cluster point of space kinetic traces associated with u . The corresponding ones associated with \tilde{u} will be denoted by $\tilde{f}_-, \tilde{n}_-, \tilde{m}_-, \tilde{f}_-^{\tau_0}$ and \tilde{f}_-^τ , respectively. We set for $(t, \bar{x}, \xi) \in \Theta_\lambda \times \mathbb{R}$,

$$F_+(t, \bar{x}, \xi) = -\mathbf{n}_1(\bar{x}_0) \cdot a(\xi) f_+^\tau(t, \bar{x}_0, \xi) + \beta'(\xi) \nabla_{\bar{x}} h(\bar{x}) \cdot \nabla_{\bar{x}} f_+^b(t, \bar{x}_0, \xi)$$

and

$$\tilde{F}_-(t, \bar{x}, \xi) = -\mathbf{n}_1(\bar{x}_0) \cdot a(\xi) \tilde{f}_-^\tau(t, \bar{x}_0, \xi) + \beta'(\xi) \nabla_{\bar{x}} h(\bar{x}) \cdot \nabla_{\bar{x}} \tilde{f}_-^b(t, \bar{x}_0, \xi)$$

where $\tilde{f}_-^b = \operatorname{sgn}^-(\tilde{u}_b - \xi)$. For $\rho, \nu \in \mathbb{R}_+$ and $\varepsilon = (\bar{\varepsilon}, \varepsilon_d) \in \mathbb{R}_+^d$, set

$$f_+^{\rho, \nu, \varepsilon} = (f_+ \times \mathbf{1}_{Q_\lambda}) * \gamma_{\rho, \nu, \varepsilon}, \quad f_+^{\tau_0, \rho, \varepsilon} = (f_+^{\tau_0} \times \mathbf{1}_{\Omega_\lambda}) * \gamma_{\rho, \varepsilon},$$

$$F_+^{\rho, \nu, \varepsilon} = (F_+ \times \mathbf{1}_{\Sigma_\lambda}) * \gamma_{\rho, \nu, \varepsilon}, \quad m_+^{\rho, \nu, \varepsilon} = (m_+ \times \mathbf{1}_{Q_\lambda}) * \gamma_{\rho, \nu, \varepsilon}$$

$$\text{and } n_+^{\rho, \nu, \varepsilon} = (n_+ \times \mathbf{1}_{Q_\lambda}) * \gamma_{\rho, \nu, \varepsilon}.$$

K. Kobayasi

As for \tilde{f}_- , $\tilde{f}_-^{\tau_0}$, \tilde{F}_- , etc., their regularizations $\tilde{f}_-^{\eta,\mu,\delta}$, $\tilde{f}_-^{\tau_0\eta,\delta}$, $\tilde{F}_-^{\eta,\mu,\delta}$, etc. are similarly defined in the same manner as above, but with different parameters η, μ, δ . Let $\psi \in C_c^\infty([0, T) \times \mathbb{R}^{d+1})^+$ and apply (2.2) in Lemma 2.1 to the test function $\psi^\lambda * \tilde{\gamma}_{\rho,\nu,\varepsilon}$, where $\tilde{\gamma}_{\rho,\nu,\varepsilon}$ is defined by $\tilde{\gamma}_{\rho,\nu,\varepsilon}(t, x, \xi) = \gamma_{\rho,\nu,\varepsilon}(-t, -x, -\xi)$:

$$\begin{aligned} & \int_{\mathbb{R}^{d+2}} \left(f_+^{\rho,\nu,\varepsilon} (\partial_t + a \cdot \nabla) \psi^\lambda - \beta' \nabla f_+^{\rho,\nu,\varepsilon} \cdot \nabla \psi^\lambda \right. \\ & \quad \left. + (f_+^{\tau_0\rho,\varepsilon} \theta_{\rho,\nu} + F_+^{\rho,\nu,\varepsilon}) \psi^\lambda \right) d\xi dt dx \\ & \geq \int_{\mathbb{R}^{d+2}} \partial_\xi \psi^\lambda d(m_+^{\rho,\nu,\varepsilon} + n_+^{\rho,\nu,\varepsilon}). \end{aligned} \quad (2.6)$$

On the other hand we can regularize the equation satisfied by \tilde{f}_- by the same method and obtain for same ψ 's,

$$\begin{aligned} & - \int_{\mathbb{R}^{d+2}} \left(\tilde{f}_-^{\eta,\mu,\delta} (\partial_t + a \cdot \nabla) \psi^\lambda + \beta' \nabla \tilde{f}_-^{\eta,\mu,\delta} \cdot \nabla \psi^\lambda \right. \\ & \quad \left. + (\tilde{f}_-^{\tau_0\eta,\delta} \theta_{\eta,\mu} + \tilde{F}_-^{\eta,\mu,\delta}) \psi^\lambda \right) d\xi dt dx \\ & \geq - \int_{\mathbb{R}^{d+2}} \partial_\xi \psi^\lambda d(\tilde{m}_-^{\eta,\mu,\delta} + \tilde{n}_-^{\eta,\mu,\delta}). \end{aligned} \quad (2.7)$$

Now let us fix a test function $\varphi(t, x) \in C_c^\infty([0, T) \times \mathbb{R}^d)^+$. Apply (2.6) to $\psi = -\tilde{f}_-^{\eta,\mu,\delta}(t, x, \xi)\varphi(t, x)$ and (2.7) to $\psi = f_+^{\rho,\nu,\varepsilon}(t, x, \xi)\varphi(t, x)$, and add the two equations together. After integrating by parts the left hand side of the resultant inequality, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^{d+2}} \left(-f_+^{\rho,\nu,\varepsilon} \tilde{f}_-^{\eta,\mu,\delta} (\partial_t + a \cdot \nabla + \beta' \Delta + 2\beta' \nabla f_+^{\rho,\nu,\varepsilon} \cdot \nabla \tilde{f}_-^{\eta,\mu,\delta}) \varphi^\lambda d\xi dt dx \right. \\ & \quad - \int_{\mathbb{R}^{d+2}} \left(f_+^{\tau_0\rho,\varepsilon} \theta_{\rho,\nu} \tilde{f}_-^{\eta,\mu,\delta} + \tilde{f}_-^{\tau_0\eta,\delta} \theta_{\eta,\mu} f_+^{\rho,\nu,\varepsilon} \right. \\ & \quad \left. \left. + F_+^{\rho,\nu,\varepsilon} \tilde{f}_-^{\eta,\mu,\delta} + \tilde{F}_-^{\eta,\mu,\delta} f_+^{\rho,\nu,\varepsilon} \right) \varphi^\lambda d\xi dt dx \right) \\ & \geq - \int_{\mathbb{R}^{d+2}} \partial_\xi \tilde{f}_-^{\eta,\mu,\delta} \varphi^\lambda d(m_+^{\rho,\nu,\varepsilon} + n_+^{\rho,\nu,\varepsilon}) - \int_{\mathbb{R}^{d+2}} \partial_\xi f_+^{\rho,\nu,\varepsilon} \varphi^\lambda d(\tilde{m}_+^{\eta,\mu,\delta} + \tilde{n}_+^{\eta,\mu,\delta}) \end{aligned}$$

Notice that if $\xi \in F$, then $f_+(t, x, \xi) = \text{sgn}^+(\beta(u(t, x)) - \beta(\xi))$ and hence $\nabla f_+^{\rho,\nu,\varepsilon} = [\delta(\xi - u) \nabla \beta(u)]^{\rho,\nu,\varepsilon} \equiv \delta(\xi - u) \times \mathbf{1}_Q * \gamma_{\rho,\nu,\varepsilon}$. Similarly, we have $\nabla \tilde{f}_-^{\eta,\mu,\delta} = [\delta(\xi - \tilde{u}) \nabla \beta(\tilde{u})]^{\eta,\mu,\delta}$. On the other hand, it is easy to see that $\partial_\xi f_+^{\rho,\nu,\varepsilon} = -\delta(\xi - u)^{\rho,\nu,\varepsilon} \equiv -[\delta(\xi - u) \times \mathbf{1}_Q] * \gamma_{\rho,\nu,\varepsilon}$ and $\partial_\xi \tilde{f}_-^{\eta,\mu,\delta} = -\delta(\xi - \tilde{u})^{\eta,\mu,\delta}$. Noting also that m_+ and \tilde{m}_- are nonnegative

measures, we have

$$\begin{aligned}
& \int_{\mathbb{R}^{d+2}} \left(-f_+^{\rho,\nu,\varepsilon} \tilde{f}_-^{\eta,\mu,\delta} (\partial_t + a \cdot \nabla + \beta' \Delta \right. \\
& \quad \left. + 2\beta' [\delta(\xi - u) \nabla \beta(u)]^{\rho,\nu,\varepsilon} [\delta(\xi - \tilde{u}) \nabla \beta(\tilde{u})]^{\eta,\mu,\delta} \right) \varphi^\lambda d\xi dt dx \\
& - \int_{\mathbb{R}^{d+2}} \left(f_+^{\tau_0\rho,\varepsilon} \theta_{\rho,\nu} \tilde{f}_-^{\eta,\mu,\delta} + \tilde{f}_-^{\tau_0\eta,\delta} \theta_{\eta,\mu} f_+^{\rho,\nu,\varepsilon} \right. \\
& \quad \left. + F_+^{\rho,\nu,\varepsilon} \tilde{f}_-^{\eta,\mu,\delta} + \tilde{F}_-^{\eta,\mu,\delta} f_+^{\rho,\nu,\varepsilon} \right) \varphi^\lambda d\xi dt dx \\
& \geq \int_{\mathbb{R}^{d+2}} \delta(\xi - \tilde{u})^{\eta,\mu,\delta} \varphi^\lambda dn_+^{\rho,\nu,\varepsilon} + \int_{\mathbb{R}^{d+2}} \delta(\xi - u)^{\rho,\nu,\varepsilon} \varphi^\lambda d\tilde{n}_-^{\eta,\mu,\delta}
\end{aligned}$$

Let successively $\eta, \mu, \bar{\delta}$ and δ_d go to $+0$:

$$\begin{aligned}
& \int_{Q_\lambda \times \mathbb{R}} \left(-f_+^{\rho,\nu,\varepsilon} \tilde{f}_- (\partial_t + a \cdot \nabla + \beta' \Delta \right. \\
& \quad \left. + 2\beta' [\delta(\xi - u) \nabla \beta(u)]^{\rho,\nu,\varepsilon} \delta(\xi - \tilde{u}) \nabla \beta(\tilde{u}) \right) \varphi^\lambda d\xi dt dx \\
& - \int_{Q_\lambda \times \mathbb{R}} \left(f_+^{\tau_0\rho,\varepsilon} \theta_{\rho,\nu} \tilde{f}_- + F_+^{\rho,\nu,\varepsilon} \tilde{f}_- \right) \varphi^\lambda d\xi dt dx \\
& \geq \int_{Q_\lambda \times \mathbb{R}} \delta(\xi - \tilde{u}) \varphi^\lambda dn_+^{\rho,\nu,\varepsilon} + \int_{Q_\lambda \times \mathbb{R}} \delta(\xi - u)^{\rho,\nu,\varepsilon} \varphi^\lambda d\tilde{n}_-.
\end{aligned}$$

Here we used the fact that regularized functions equal zero at $t = 0$ and at the boundary. Then, let successively $\rho, \nu, \bar{\varepsilon}$ and ε_d go to $+0$ and use (2.2) in Lemma 2.1 to obtain

$$\begin{aligned}
& \int_{Q_\lambda \times \mathbb{R}} \left(-f_+ \tilde{f}_- (\partial_t + a \cdot \nabla + \beta' \Delta \right. \\
& \quad \left. + 2\beta' \delta(\xi - u) \delta(\xi - \tilde{u}) \nabla \beta(u) \cdot \nabla \beta(\tilde{u}) \right) \varphi^\lambda d\xi dt dx \\
& - \int_{\Omega_\lambda \times \mathbb{R}} f_+^{\tau_0} \tilde{f}_-^{\tau_0} \varphi^\lambda(0, \cdot) d\xi dx + \int_{\Sigma_\lambda \times \mathbb{R}} \left((\mathbf{n}_1 \cdot a) f_+^\tau \tilde{f}_-^\tau \right. \\
& \quad \left. - \beta' (\nabla_{\bar{x}} h \cdot \nabla_{\bar{x}} f_+^b) \tilde{f}_-^b \right) \bar{\varphi}^\lambda d\xi dt d\bar{x} \\
& \geq \int_{Q_\lambda \times \mathbb{R}} \delta(\xi - \tilde{u}) \varphi^\lambda dn_+ + \int_{Q_\lambda \times \mathbb{R}} \delta(\xi - u) \varphi^\lambda d\tilde{n}_-.
\end{aligned} \tag{2.8}$$

Next, let successively $\rho, \nu, \bar{\varepsilon}$ and ε_d go to $+0$ and then let successively $\eta, \mu, \bar{\delta}$ and δ_d go to $+0$: For any weak* cluster point \tilde{f}_-^τ and for some

K. Kobayasi

weak* cluster point f_+^τ , we have

$$\begin{aligned}
& \int_{Q_\lambda \times \mathbb{R}} \left(-f_+ \tilde{f}_- (\partial_t + a \cdot \nabla + \beta' \Delta \right. \\
& \quad \left. + 2\beta' \delta(\xi - u) \delta(\xi - \tilde{u}) \nabla \beta(u) \cdot \nabla \beta(\tilde{u})) \varphi^\lambda d\xi dt dx \right. \\
& \quad \left. - \int_{\Omega_\lambda \times \mathbb{R}} f_+^{\tau_0} \tilde{f}_-^{\tau_0} \varphi^\lambda(0, \cdot) d\xi dx \right. \\
& \quad \left. + \int_{\Sigma_\lambda \times \mathbb{R}} \left((\mathbf{n}_1 \cdot a) f_+^\tau \tilde{f}_-^\tau - \beta' (\nabla_{\bar{x}} h \cdot \nabla_{\bar{x}} \tilde{f}_-^b) f_+^b \right) \bar{\varphi}^\lambda d\xi dt d\bar{x} \right) \\
& \geq \int_{Q_\lambda \times \mathbb{R}} \delta(\xi - \tilde{u}) \varphi^\lambda dn_+ + \int_{Q_\lambda \times \mathbb{R}} \delta(\xi - u) \varphi^\lambda d\tilde{n}_-.
\end{aligned} \tag{2.9}$$

Adding (2.8) and (2.9) yields

$$\begin{aligned}
& \int_{Q_\lambda \times \mathbb{R}} \left(-f_+ \tilde{f}_- (\partial_t + a \cdot \nabla + \beta' \Delta \right. \\
& \quad \left. + 2\beta' \delta(\xi - u) \delta(\xi - \tilde{u}) \nabla \beta(u) \cdot \nabla \beta(\tilde{u})) \varphi^\lambda d\xi dt dx \right. \\
& \quad \left. - \int_{\Omega_\lambda \times \mathbb{R}} f_+^{\tau_0} \tilde{f}_-^{\tau_0} \varphi^\lambda(0, \cdot) d\xi dx \right. \\
& \quad \left. + \int_{\Sigma_\lambda \times \mathbb{R}} \left((\mathbf{n}_1 \cdot a) f_+^\tau \tilde{f}_-^\tau - \frac{1}{2} \beta' \nabla_{\bar{x}} h \cdot \nabla_{\bar{x}} (f_+^b \tilde{f}_-^b) \right) \bar{\varphi}^\lambda d\xi dt d\bar{x} \right) \\
& \geq \int_{Q_\lambda \times \mathbb{R}} (\delta(\xi - \tilde{u}) n_+ + \delta(\xi - u) \tilde{n}_-) \varphi^\lambda d\xi dt dx.
\end{aligned}$$

for some weak* cluster points f_+^τ and \tilde{f}_-^τ . Since

$$\begin{aligned}
& 2\beta'(\xi) \delta(\xi - u) \delta(\xi - \tilde{u}) \nabla \beta(u) \cdot \nabla \beta(\tilde{u}) \\
& \leq 1_F(\xi) \delta(\xi - u) \delta(\xi - \tilde{u}) (|\nabla \beta(u)|^2 + |\nabla \beta(\tilde{u})|^2) \\
& = \delta(\xi - \tilde{u}) n_+(t, x, \xi) + \delta(\xi - u) \tilde{n}_-(t, x, \xi),
\end{aligned}$$

we arrive at

$$\begin{aligned}
& - \int_{Q_\lambda \times \mathbb{R}} f_+ \tilde{f}_- (\partial_t + a \cdot \nabla + \beta' \Delta) \varphi^\lambda d\xi dt dx \\
& \geq \int_{\Omega_\lambda \times \mathbb{R}} f_+^{\tau_0} \tilde{f}_-^{\tau_0} \varphi^\lambda(0, \cdot) d\xi dx - \int_{\Sigma_\lambda \times \mathbb{R}} \left((\mathbf{n}_1 \cdot a) f_+^\tau \tilde{f}_-^\tau \right. \\
& \quad \left. - \frac{1}{2} \beta' \nabla_{\bar{x}} h \cdot \nabla_{\bar{x}} (f_+^b \tilde{f}_-^b) \right) \bar{\varphi}^\lambda d\xi dt d\bar{x}.
\end{aligned} \tag{2.10}$$

We compute each term of (2.26). Firstly,

$$\begin{aligned} & - \int_{Q_\lambda \times \mathbb{R}} f_+ \tilde{f}_- (\partial_t + a \cdot \nabla + \beta' \Delta) \varphi^\lambda d\xi dt dx \\ & = \int_{Q_\lambda} \left((u - \tilde{u})^+ + \mathcal{F}^+(u, \tilde{u}) \nabla \varphi^\lambda + (\beta(u) - \beta(\tilde{u}))^+ \Delta \varphi^\lambda \right) dt dx. \end{aligned} \quad (2.11)$$

Secondly, by virtue of Lemma 2.2 and by using integration by parts one can calculate:

$$\begin{aligned} & \int_{\mathbb{R}} f_+^{\tau_0} \tilde{f}_-^{\tau_0} d\xi \\ & = \int_{-\infty}^{\tilde{u}_0} \nu_x^{\tau_0}([\xi, \infty)) \partial_\xi \tilde{m}_-^0 d\xi - \int_{\tilde{u}_0}^{u_0 \vee \tilde{u}_0} \nu_x^{\tau_0}([\xi, \infty)) \tilde{\nu}_x^{\tau_0}((-\infty, \xi]) d\xi \\ & \quad - \int_{u_0 \vee \tilde{u}_0}^{\infty} \partial_\xi m_+^0 \tilde{\nu}_x^{\tau_0}((-\infty, \xi]) d\xi \\ & = \nu_x^{\tau_0}([\tilde{u}_0, \infty)) \tilde{m}_-^0(\cdot, \tilde{u}_0) + \int_{-\infty}^{\tilde{u}_0} \tilde{m}_-^0 d\nu_x^{\tau_0} - \int_{\tilde{u}_0}^{u_0 \vee \tilde{u}_0} \nu_x^{\tau_0}([\xi, \infty)) \tilde{\nu}_x^{\tau_0}((-\infty, \xi]) d\xi \\ & \quad + m_+^0(\cdot, u_0 \vee \tilde{u}_0) \tilde{\nu}_x^{\tau_0}((-\infty, u_0 \vee \tilde{u}_0]) + \int_{u_0 \vee \tilde{u}_0}^{\infty} m_+^0 d\tilde{\nu}_+^{\tau_0} \\ & \geq - \int_{\tilde{u}_0}^{u_0 \vee \tilde{u}_0} d\xi = -(u - \tilde{u}_0)^+. \end{aligned}$$

Here we used the fact that $d\nu_x^{\tau_0}([\xi, \infty))/d\xi = -d\nu_x^{\tau_0}(\xi)$ and $d\tilde{\nu}_-^{\tau_0}((-\infty, \xi])/d\xi = d\tilde{\nu}_-^{\tau_0}(\xi)$. Thus we have

$$\int_{\Omega_\lambda \times \mathbb{R}} f_+^{\tau_0} \tilde{f}_-^{\tau_0} \varphi^\lambda(0, \cdot) d\xi dx \geq - \int_{\Omega_\lambda} (u_0 - \tilde{u}_0)^+ \varphi^\lambda(0, \cdot) dx. \quad (2.12)$$

Finally, we calculate analogously the boundary term by using Lemma 2.4:

$$\begin{aligned} & \int_{\mathbb{R}} (\mathbf{n}_1 \cdot a) f_+^\tau \tilde{f}_-^\tau d\xi \\ & = - \int_{-\infty}^{\tilde{u}_b} \partial_\xi \tilde{m}_-^b \nu_{t,y}^\tau([\xi, \infty)) d\xi - \int_{\tilde{u}_b}^{u_b \vee \tilde{u}_b} (\mathbf{n}_1 \cdot a) \nu_{t,y}^\tau([\xi, \infty)) \tilde{\nu}_{t,y}^\tau((-\infty, \xi]) d\xi \\ & \quad + \int_{u_b \vee \tilde{u}_b}^{\infty} \partial_\xi m_+^b \nu_{t,y}^\tau((-\infty, \xi]) d\xi \\ & \leq M \int_{\tilde{u}_b}^{u_b \vee \tilde{u}_b} d\xi = M(u_b - \tilde{u}_b)^+, \end{aligned}$$

where y stands for \bar{x}_0 and we used the fact that $d\nu_{t,y}^\tau([\xi, \infty))/d\xi = -d\nu_{t,y}^\tau(\xi)$ and $d\tilde{\nu}_{t,y}^\tau((-\infty, \xi])/d\xi = d\tilde{\nu}_{t,y}^\tau(\xi)$ as well as the fact that $m_+^b \geq$

K. Kobayasi

0 for $\xi \geq u_b$ and $\tilde{m}_-^b \geq 0$ for $\xi \leq \tilde{u}_b$ by virtue of (2.21) and the corresponding inequality associated with \tilde{u} , respectively. This implies

$$\int_{\Sigma_\lambda \times \mathbb{R}} (\mathbf{n}_1 \cdot a) f_+^\tau \tilde{f}_-^\tau \bar{\varphi}^\lambda d\xi dt d\bar{x} \leq M \int_{\Sigma_\lambda} (u_b - \tilde{u}_b)^+ \bar{\varphi}^\lambda dt d\bar{x}. \quad (2.13)$$

Moreover

$$\begin{aligned} & \int_{\Sigma_\lambda \times \mathbb{R}} \beta'(\xi) \nabla_{\bar{x}} h(\bar{x}) \cdot \nabla_{\bar{x}} (f_+^b \tilde{f}_-^b) \bar{\varphi}^\lambda d\xi dt d\bar{x} \\ &= - \int_{\Sigma_\lambda \times \mathbb{R}} \beta'(\xi) \operatorname{div}_{\bar{x}} (\bar{\varphi}^\lambda \nabla_{\bar{x}} h(\bar{x})) f_+^b \tilde{f}_-^b d\xi dt d\bar{x} \\ &= - \int_{\Sigma_\lambda} (\beta(u_b) - \beta(\tilde{u}_b))^+ (\Delta_{\bar{x}} h(\bar{x}) \bar{\varphi}^\lambda + \nabla_{\bar{x}} h(\bar{x}) \cdot \nabla_{\bar{x}} \bar{\varphi}^\lambda) dt d\bar{x} \\ &\geq \int_{\Sigma_\lambda} (\beta(u_b) - \beta(\tilde{u}_b))^+ (-L \bar{\varphi}^\lambda + \nabla_{\bar{x}} h(\bar{x}) \cdot \nabla_{\bar{x}} \bar{\varphi}^\lambda) dt d\bar{x}. \end{aligned} \quad (2.14)$$

Combining (2.10) with (2.11) through (2.14) and choosing appropriate test functions φ 's, we arrive at the estimate

$$\begin{aligned} & \frac{1}{T} \int_{Q_\lambda} (u - \tilde{u})^+ dt dx \\ &\leq \int_{\Omega_\lambda} (u_0 - \tilde{u}_0)^+ dx + M \int_{\Sigma_\lambda} (u_b - \tilde{u}_b)^+ dt d\bar{x} + \frac{L}{2} \int_{\Sigma_\lambda} (\beta(u_b) - \beta(\tilde{u}_b))^+ dt d\bar{x}. \end{aligned}$$

By summing over $i = 0, 1, \dots, N$, we obtain the desired estimate (1.9) and the proof of Theorem is complete.

REFERENCES

- [1] C. Bardos, A. Y. LeRoux and J.-C. Nédélec, First order quasilinear equations with boundary conditions, *Comm. Partial Differential Equations*, 4 (1979), pp. 1017–1034.
- [2] J. Carrillo, On the uniqueness of the solution of the evolution dam problem, *Nonlinear anal.*, 22 (1994), pp. 573–607.
- [3] J. Carrillo, Entropy solutions for nonlinear degenerate problems, *Arch. Rational Mech. Anal.* 147 (1999), pp. 269–361.
- [4] G. -Q. Chen and H. Frid, Divergence-Measure fields and the Euler equations for gas dynamics, *Commun. Math. Phys.* 236 (2003), pp. 251–280.
- [5] J. I. Diaz and R. Kershner, On a nonlinear degenerate parabolic equation on filtration or evaporation through a porous medium, *J. Differential Equations* 69 (1987), pp. 368–403.
- [6] J. Droniou, C. Imbert and J. Vovelle, An error estimate for the parabolic approximation of multidimensional scalar conservation laws with boundary conditions, *Ann. I. H. Poincaré* 21(2004), pp. 689–714.
- [7] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, CRC Press Inc., Boca Raton, Ann Arbor, London, 1992

- [8] G. Gagneux and M. Madaune-Tort, *Analyse mathématique de modèles non linéaires de l'ingénierie pétrolière*, Mathématique and Applications 22, Springer-Verlag, Berlin, Heidelberg, New York, 1996.
- [9] C. Imbert and J. Vovelle, A kinetic formulation for multidimensional scalar conservation laws with boundary conditions and applications, *SIAM J. Math. Anal.*, 36 (2004), pp. 214–232.
- [10] K. Kobayasi, The equivalence of weak solutions and entropy solutions of nonlinear degenerate second - order equations, *J. Differential Equations*, 189 (2003), pp. 383–395.
- [11] S. N. Kružkov, First order quasilinear equations with several independent variables, *Mat. Sbornik*, 81 (1970), pp. 228–255.
- [12] P. L. Lions, B. Perthame and E. Tadmor, A kinetic formulation of mutidimensional scalar conservation laws and related equations, *J. Amer. Math. Soc.*, 7 (1994), pp. 169–191.
- [13] J. Málek, J. Nečas, M. Rokyta and M. Ružička, *Weak and measure-valued solutions to evolutionary PDEs*, Chapman & Hall, London, 1996.
- [14] C. Mascia, A. Porretta and A. Terracina, Nonhomogeneous Dirichlet problems for degenerate parabolic - hyperbolic equations, *Arch. Rational Mech. Anal.*, 163 (2002), pp. 87–124.
- [15] A. Michel and J. Vovelle, Entropy formulation for parabolic degenerate equations with general Dirichlet boundary conditions and application to the convergence of FV methods, *SIAM Numer. Anal.*, 41 (2003), pp. 2262–2293.
- [16] F. Otto, Initial-boundary value for a scalar conservation laws, *C. R. Acad. Sci. Paris Sér I Math.*, 322 (1996), pp. 729–734.
- [17] B. Perthame, Uniqueness and error estimates in first order quasilinear conservation laws via the kinetic entropy defect measure, *J. Math. Pures Appl.*, 77 (1998), pp. 1055–1064.
- [18] B. parthame, *Kinetic formulation of conservation laws*, Oxford Univ. Press, Oxford, UK, 2002.
- [19] A. Vasseur, Strong traces for solutions of multidimensinal scalar conservation laws, *Arch. Rational Mech. Anal.*, 160 (2001), pp. 181–193.